

NUMBERS FOR EVERYDAY LIFE

Real Number and Numeration System

INTRODUCTION

In this section, you will learn more about the subsets of the real number system and basic operations on them. Specifically, you will study surds, exponents (indices), logarithms and modular arithmetic. These are fundamental concepts in mathematics which have various applications in everyday life. Surds have applications in physics, number theory and finance. Indices and logarithms are applied in banking, population growth models and engineering. Modular arithmetic is used to design calendar systems and duty rosters. By the end of this section, you will be able to evaluate the relationships between the laws and properties of surds, indices and logarithms and apply them to solve problems in everyday life.

KEY IDEAS

- **Applications of logarithms:** Logarithms have several applications in real life.
- **Concept of logarithms:** Logarithm is the inverse of indices. That is, logarithm can be used to reverse indices.
- **Indices in which the power is an integer:** If the power of an indicial expression is an integer, three situations arise; the integer may be positive, zero or negative.
- **Keywords:** Pure surds, mixed surds, compound surds, radical, radicand, rationalisation, conjugate, conjugate pair, exponents, indices, logarithm, base, augment.
- **Laws of logarithms:** Like indices, there are laws, which guide us to perform operations on logarithms.
- **Operations on surds:** This involves the addition, subtraction, multiplication and division of surds.

- **Order of operations involving indices:** when several operations are involved in a computation, there are rules that will help you to do the computation.
- **Perfect squares:** A perfect square is a number whose square root is a natural number. In other words, when you square a natural number, you get a perfect square.
- **Rationalisation:** This is the process of making the denominator of a given surd rational.
- **Rules of indices:** There are rules that will guide you to perform operations on indices.
- **Simplification of surds:** This is the process of expressing a surd in its simplest form.
- **Surd or radical:** The square root of a number, which is not a perfect square, gives rise to a surd, also known as a radical. These are irrational numbers.

INTRODUCTION AND SIMPLIFICATION OF SURDS

Do you remember the set of perfect squares you learnt in JHS? Write down at least six perfect squares and show them to a classmate. The set of perfect squares $P = \{1, 4, 9, 16, 25, 36, 49 \dots\}$. When we find the square root of a number, which is **not** a perfect square, we get a surd. Examples of surds include $\sqrt{2}$, $\sqrt{3}$, $-2\sqrt{5}$ and $\frac{2 - \sqrt{2}}{3}$.

Perform the activity below with in pairs or in a small group. This activity will help you to generate and simplify surds.

Activity 1.1: Generating Surds Using the Geodot or Graph Paper

Materials needed: Geodot or graph paper, pencils and rulers.

Step 1: Take each small square of four dots on the geodot as a unit square. If you are using a graph paper, take a one-centimetre square as a unit square.

Draw a unit square on your graph or geodot paper as shown below.

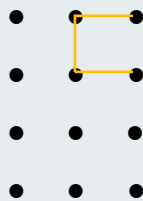


Figure 1.1: Drawing unit square on geodot

Step 2: Draw a two-unit square on the graph or geodot paper. Count the number of unit squares in the square you have drawn. If you counted four-unit squares, you are correct.

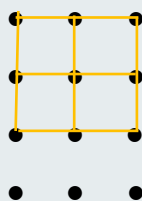


Figure 1.2: Drawing two-unit squares on geodot

Step 3: Draw a three-unit square on your graph or geodot paper and count the number of unit squares in it.

Step 4: If you draw a six-unit square, how many unit squares will you find in it? What are your observations? Share your observations with your group.

Step 5: Now, draw a rectangle on your geodot or graph paper and count the number of unit squares in the rectangle you have drawn. Is the number you obtained a perfect square? What conclusion can you draw from this activity? Share your ideas with your group.

You can see that the first four steps of this activity assist you to generate the set of perfect squares. The fifth step gives you an example of a non-perfect square.

If you find the square root of non-perfect squares, you will get surds or radicals like $\sqrt{2}$, $\sqrt{3}$ and so on. Note that because 1, 4 and 9 are perfect squares, $\sqrt{1} = \sqrt{1 \times 1} = 1$, $\sqrt{4} = \sqrt{2 \times 2} = 2$ and $\sqrt{9} = \sqrt{3 \times 3} = 3$.

Activity 1.2: Simplifying Surds Using the Geodot or Graph Paper

Materials needed: Geodot or graph paper, pencils and rulers.

To simplify $\sqrt{12}$, using the geodot, draw a rectangle with 12 dots as shown in the diagram below. What is the largest possible square you can draw within

the space of the 12 dots? How many unit squares are in the square you have drawn?

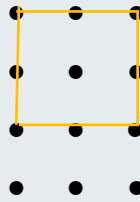


Figure 1.3: Drawing 4-unit squares on geodot

The square in the geodot has 4-unit squares within it. Thus, you can write $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$.

Similarly, to simplify $\sqrt{24}$, express 24 as a product of two numbers, one of which must be a perfect square. That is, $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2 \times \sqrt{6} = 2\sqrt{6}$.

Also, $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2} = 4\sqrt{2}$.

Activity 1.3: Finding the Square Root of Perfect Squares Using Prime Factors

The square root of a perfect square can be found by breaking the number down into its prime factors. If the number has an even number of prime factors, we can find the square root easily as shown in the examples below.

Example 1.1

$$\sqrt{169} = \sqrt{13 \times 13} = 13.$$

Example 1.2

$$\sqrt{36} = \sqrt{2 \times 2 \times 3 \times 3} = \sqrt{2 \times 2} \times \sqrt{3 \times 3} = 2 \times 3 = 6.$$

Example 1.3

$$\sqrt{100} = \sqrt{2 \times 2 \times 5 \times 5} = \sqrt{2 \times 2} \times \sqrt{5 \times 5} = 2 \times 5 = 10.$$

There is yet another method for finding the square root of perfect squares. You will be exposed to this method in the following activity.

Activity 1.4: Finding the Square Root of Perfect Squares Using Repeated Subtraction

To use this method to find the square root of a perfect square, list the set of odd numbers and follow the steps below.

Step 1: Subtract the first odd number from the perfect square.

Step 2: Subtract the second odd number from the result of step 1.

Step 3: Subtract the third odd number from the result of step 2.

Step 4: Do this until you get zero.

Step 5: The number of times we subtract an odd number to get zero is the square root of the perfect square.

Example 1.4

Find the square root of 36, using the repeated subtraction method.

1. $36 - 1 = 35$
2. $35 - 3 = 32$
3. $32 - 5 = 27$
4. $27 - 7 = 20$
5. $20 - 9 = 11$
6. $11 - 11 = 0$

Since the subtraction was done six times to get zero, the square root of 36 is 6.

Example 1.5

Find the square root of 49, using the repeated subtraction method.

1. $49 - 1 = 48$
2. $48 - 3 = 45$
3. $45 - 5 = 40$
4. $40 - 7 = 33$
5. $33 - 9 = 24$
6. $24 - 11 = 13$
7. $13 - 13 = 0$

Because it took seven steps to get zero, $\sqrt{49} = 7$.

OPERATIONS WITH SURDS AND THEIR APPLICATIONS

In this lesson, you are going to perform the basic operations on surds. That is, you are going to learn how to add, subtract, multiply and divide surds.

Activity 1.5: Addition and subtraction of surds

You can perform this activity alone or with a partner.

Materials needed: cardboard, board markers and a pair of scissors.

Step 1: Cut out about 20 pieces from the cardboard, making sure the cards you cut out have almost the same shape and size.

Step 2: Write $\sqrt{3}$ on ten of the cards and $\sqrt{6}$ on the remaining pieces.

Step 3: Take some of the cards with $\sqrt{6}$ on them. Ask your partner to take some of the cards with $\sqrt{6}$ on them. How many cards do you have? How will you represent that on paper? How many pieces does your partner have? How will you write what your partner has on paper?

Now, how many cards do you and your partner have altogether? How will you represent the operation you have just performed in mathematical symbols? For example, it could be something like:

$$\begin{bmatrix} \sqrt{6} \\ \sqrt{6} \\ \sqrt{6} \end{bmatrix} + \begin{bmatrix} \sqrt{6} & \sqrt{6} \\ \sqrt{6} & \sqrt{6} \\ \sqrt{6} & \sqrt{6} \end{bmatrix} = 3\sqrt{6} + 6\sqrt{6} = 9\sqrt{6}$$

Step 4: Take some of the $\sqrt{3}$ cards. How many cards did you pick? Give some of your cards to your partner. How many do you have left? How will you represent the operation you have just performed in mathematical symbols?

Step 5: Take some of the cards with $\sqrt{3}$ on them. Let your friend take some of the cards with $\sqrt{6}$ on them. How many cards do you have in total? How will you represent the operation you have just performed on a piece of paper? Show it to your friend.

In Activity 1.5, you have learnt that we can add or subtract like surds (surds with the same radicand) by counting forwards or backwards. However, unlike surds cannot be added or subtracted.

In general, if x and y are rational numbers and z is a non-perfect square,

a $x\sqrt{z} + y\sqrt{z} = (x + y)\sqrt{z}$

b $x\sqrt{z} - y\sqrt{z} = (x - y)\sqrt{z}$

c $x\sqrt{a} \pm y\sqrt{b} = x\sqrt{a} \pm y\sqrt{b}$

Example 1.8

Simplify the following:

1. $3\sqrt{6} + 6\sqrt{6}$

2. $12\sqrt{3} + 6\sqrt{3}$

3. $3\sqrt{6} - 6\sqrt{6}$

Solution

1. $3\sqrt{6} + 6\sqrt{6} = (3 + 6)\sqrt{6} = 9\sqrt{6}$

2. $12\sqrt{3} + 6\sqrt{3} = (12 + 6)\sqrt{3} = 18\sqrt{3}$

3. $3\sqrt{6} - 6\sqrt{6} = (3 - 6)\sqrt{6} = -3\sqrt{6}$

Example 1.9

Simplify $3\sqrt{8} + 2\sqrt{18} - 6\sqrt{2}$

Solution

$$\begin{aligned} & 3\sqrt{8} + 2\sqrt{18} - 6\sqrt{2} \\ &= 3\sqrt{4 \times 2} + 2\sqrt{9 \times 2} - 6\sqrt{2} \\ &= 3\sqrt{4} \times \sqrt{2} + 2\sqrt{9} \times \sqrt{2} - 6\sqrt{2} \\ &= 3 \times 2 \times \sqrt{2} + 2 \times 3 \times \sqrt{2} - 6\sqrt{2} \\ &= 6\sqrt{2} + 6\sqrt{2} - 6\sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

Example 1.10

Simplify $2\sqrt{24} + 3\sqrt{32} - 9\sqrt{2}$

Solution

$$\begin{aligned} & 2\sqrt{24} + 3\sqrt{32} - 9\sqrt{2} \\ &= 2\sqrt{4 \times 6} + 3\sqrt{16 \times 2} - 9\sqrt{2} \\ &= 2\sqrt{4} \times \sqrt{6} + 3\sqrt{16} \times \sqrt{2} - 9\sqrt{2} \end{aligned}$$

$$\begin{aligned}
&= 2 \times 2 \times \sqrt{6} + 3 \times 4 \times \sqrt{2} - 9\sqrt{2} \\
&= 4\sqrt{6} + 12\sqrt{2} - 9\sqrt{2} \\
&= 4\sqrt{6} + 3\sqrt{2}
\end{aligned}$$

Activity 1.6: Multiplication of surds

Whole class activity

Materials needed: tables and chairs.

Step 1: Place three tables and three chairs in front of the class. Arrange the tables and chairs in such a way that if someone is facing the board, the chair will be on the left and the table will be on the right. Leave about a stride between the first table and chair, the second table and chair and the third table and chair.

Step 2: Two classmates to sit on the chairs and two others to squat under the first two tables.

Step 3: Ask the friends sitting on the chairs to walk together to sit on the third chair and let the friends squatting under the table walk together to squat under the third table.

This activity demonstrates that when multiplying two surds the numbers outside the square root sign multiply each other and the numbers under the square root sign multiply each other.

For example, $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.

Note that two surds can be multiplied whether they are like or not.

Example 1.11

Express the following in the simplest form:

1. $2\sqrt{3} \times 5\sqrt{2} = 2 \times 5 \sqrt{3 \times 2} = 10\sqrt{6}$
2. $6\sqrt{2} \times 2\sqrt{2} = 6 \times 2 \sqrt{2 \times 2} = 12\sqrt{4} = 12 \times 2 = 24$

Division of Surds

Division of surds is similar to multiplication of surds.

For example, $\frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \sqrt{\frac{b}{d}}$.

Example 1.12

Simplify the following, leaving your answer in surd form:

1. $\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}$
2. $\frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$
3. $\frac{\sqrt{16}}{\sqrt{9}} = \sqrt{\frac{16}{9}} = \frac{\sqrt{16}}{\sqrt{9}} = \frac{4}{3} = 1\frac{1}{3}$

Rationalising Monomial Denominators

Mathematicians do not like to see an irrational number as a denominator. For example, mathematicians are uncomfortable with surds like $\frac{1}{\sqrt{2}}$, $\frac{3}{\sqrt{7}}$ and $\frac{5+\sqrt{3}}{3\sqrt{5}}$. Whenever this happens, they remove the irrational number from the denominator. This is done in a process known as rationalisation. That is, the surd expression is manipulated until the irrational denominator becomes rational.

To rationalise a surd with an irrational monomial denominator, multiply both the numerator and denominator by the same irrational expression. When you simplify this expression, you will get an expression whose denominator is a rational number.

The number you must multiply numerator and denominator by is known as the conjugate of the irrational number you want to eliminate from the denominator. The conjugate is the surd, which multiplies a given surd to make it a rational number. For example, $3\sqrt{2} \times \sqrt{2} = 3 \times \sqrt{2} \times \sqrt{2} = 3 \times 2 = 6$. Thus, the best conjugate of $3\sqrt{2}$ is $\sqrt{2}$ even though $3\sqrt{2}$ is also a conjugate. This is because $3\sqrt{2} \times 3\sqrt{2} = 9 \times 2 = 18$.

Example 1.6

Rationalise $\frac{1}{\sqrt{2}}$.

Solution

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Example 1.7

Rationalise $\frac{3}{\sqrt{7}}$.

Solution

$$\frac{3}{\sqrt{7}} = \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{3\sqrt{7}}{7}$$

BASIC CONCEPTS AND PROPERTIES OF EXPONENTS

If 2 is multiplied by itself six times, you can write $2 \times 2 \times 2 \times 2 \times 2 \times 2$. If 2 is multiplied by itself 100 times, imagine how tedious it would be to write it out. Consider the time it will take and the space it will occupy. To simplify this, when 2 is multiplied by itself 100 times, mathematicians express this as 2^{100} , read as ‘two to the power 100’. This abbreviation makes writing it far less tedious and time consuming. In the expression 2^{100} , 2 is known as the base and 100 is the power or exponent.

Powers of integers

The number to which we raise a base can be called an index (plural indices), a power or an exponent.

Positive exponents: A positive exponent, n , means that the base is multiplied by itself n times. For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ and $-3^4 = -3 \times -3 \times -3 \times -3 = 81$.

Zero exponents: Any number raised to the exponent of zero is always 1. For example, $6^0 = 1$, $-4^0 = 1$ and $\left(\frac{3}{4}\right)^0 = 1$.

Negative exponents: A negative exponent, n , means that the reciprocal of the base is raised to the absolute value of the exponent. For example, $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$, $-6^{-2} = \frac{1}{-6^2} = \frac{1}{-6 \times -6} = \frac{1}{36}$.

Scientific calculators can be used to verify answers, but if the base is negative, be sure to put it in brackets before raising to the exponent.

Order of Operations with Exponents

Even though BODMAS is usually interpreted as *bracket, of, division, multiplication, addition and subtraction, of* can also be thought of as *order*. *Order* represents roots and exponents. Thus, BODMAS implies that, moving from left to right, you have to expand the brackets first before coming to the order (roots and exponents), followed by division/ multiplication, and then addition/subtraction. That is, if mixture of these operations is found in the same expression.

For example, $3 - \sqrt{16} \div 2 = 3 - 4 \div 2 = 3 - 2 = 1$.

Or, $1 - 2 + 3 \times 4 \div 5^2 = 1 - 2 + 3 \times 4 \div 25$

$$\begin{aligned}
 &= 1 - 2 + 3 \times \frac{4}{25} \\
 &= 1 - 2 + \frac{12}{25} \\
 &= -1 + \frac{12}{25} \\
 &= -\frac{13}{25}
 \end{aligned}$$

Rules of Exponents

There are important exponent rules to remember when simplifying expressions involving indices or exponents.

They are the product rule, the quotient rule and the power of a power rule.

The Product Rule

$$6^3 \times 6^5 = (6 \times 6 \times 6) \times (6 \times 6 \times 6 \times 6 \times 6) = 6^8 = 6^{3+5}.$$

In general terms, $m^a \times m^b = m^{a+b}$.

This is the product rule which tells us that when multiplying terms with the same base you add the exponents.

For example, $3^2 \times 3^6 = 3^{6+2} = 3^8$ and $7^{-4} \times 7^6 = 7^{-4+6} = 7^2$

The Quotient Rule

$$5^7 \div 5^4 = \frac{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}{5 \times 5 \times 5 \times 5} = 5^3 = 5^{7-4}$$

In general terms, $m^a \div m^b = m^{a-b}$.

This is the quotient rule which tells us that when dividing terms with the same base you subtract the exponents.

For example, $5^3 \div 5^6 = 5^{3-6} = 5^{-3} = \frac{1}{5^3} = \frac{1}{125}$ and $9^{-4} \div 9^6 = 9^{-4-6} = 9^{-10} = \frac{1}{9^{10}}$

The Power of a Power Rule

$$(2^3)^4 = 2^3 \times 2^3 \times 2^3 \times 2^3 = 2^{3+3+3+3} = 2^{3 \times 4} = 2^{12}.$$

In general terms, $(a^m)^n = a^{mn}$.

For example, $(4^3)^2 = 4^{3 \times 2} = 4^6$ and $(3^{-1})^4 = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$.

Power of a Product

In general terms, $(ab)^m = a^m \times b^m$

For example: $(2 \times 3)^2 = 2^2 \times 3^2 = 4 \times 9 = 36$

Exponential Equations

An **exponential equation** is an equation in which the **variable appears in the exponent**.

Form:

$$a^x = b$$

or

$$a^{f(x)} = g(x)$$

These equations require **logarithms**, **common bases**, or **trial and error** to solve.

Worked Examples

Example 1

Solve:

$$2^x = 8$$

Solution

Write 8 as a power of 2:

$$8 = 2^3 \Rightarrow 2^x = 2^3 \Rightarrow x = 3$$

Example 2

Solve:

$$5^{x+1} = 125$$

Solution

$$125 = 5^3 \Rightarrow 5^{x+1} = 5^3 \Rightarrow x + 1 = 3 \Rightarrow x = 2$$

Example 3

Solve:

$$3^x = \frac{1}{27}$$

Solution

$$\frac{1}{27} = 3^{-3} \Rightarrow 3^x = 3^{-3} \Rightarrow x = -3$$

The following examples will help you to apply the rules of indices.

Example 1.13

Evaluate the following:

1. $81^{\frac{3}{4}}$
2. $8^{\frac{2}{3}}$
3. $64^{\frac{-1}{2}} \times 216^{\frac{2}{3}}$
4. $\left(\frac{27}{216}\right)^{\frac{1}{3}}$

Solution

$$1. \quad 81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}} = 3^{4 \times \frac{3}{4}} = 3^3 = 27.$$

$$2. \quad 8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^{3 \times \frac{2}{3}} = 2^2 = 4$$

$$3. \quad 64^{\frac{-1}{2}} \times 216^{\frac{2}{3}} = (2^6)^{\frac{-1}{2}} \times (6^3)^{\frac{2}{3}}$$

$$= 2^{6 \times \frac{-1}{2}} \times 6^{3 \times \frac{2}{3}}$$

$$= 2^{-3} \times 6^2$$

$$= \frac{1}{2^3} \times 36$$

$$= \frac{1}{8} \times 36$$

$$= \frac{36}{8} = 4.5$$

$$4. \quad \left(\frac{27}{216}\right)^{\frac{1}{3}} = \left(\frac{3^3}{6^3}\right)^{\frac{1}{3}}$$

$$= \left[\left(\frac{3}{6}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[\left(\frac{1}{2}\right)^3\right]^{\frac{1}{3}}$$

$$= \left(\frac{1}{2}\right)^{3 \times \frac{1}{3}}$$

$$= \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

Real-Life Applications of Indices (Exponents)

Exponents are used in **population growth**, **compound interest**, **computing**, **medicine**, and **radioactive decay**.

Example 1: Compound Interest (Banking)

You invest GH¢1,000 at an annual interest rate of 10% compounded yearly. What will the value be after 3 years?

Formula:

$$A = P(1 + r)^t$$

Where:

- $P = 1000$
- $r = 0.10$
- $t = 3$

Solution

$$\begin{aligned} A &= 1000(1 + 0.10)^3 \\ &= 1000(1.1)^3 \\ &= 1000(1.331) \\ &= \text{GH¢}1331 \end{aligned}$$

Example 2: Bacteria Growth (Biology)

A bacteria population doubles every hour. If you start with 200 bacteria, how many will there be after 4 hours?

Formula:

$$\begin{aligned} \text{Population} &= 200 \times 2^4 \\ &= 200 \times 16 \\ &= 3200 \end{aligned}$$

Example 3: Computer Memory (ICT)

Each time you upgrade your memory card, its capacity doubles. If your original card is 8 GB, what will be the capacity after 3 upgrades?

$$\text{Capacity} = 8 \times 2^3 = 8 \times 8 = 64 \text{ GB}$$

Example 4: Radioactive Decay (Physics)

A radioactive substance decays such that half remains every 5 hours. If you start with 80 grams, how much remains after 15 hours?

Solution

15 hours = 3 half-lives

$$\text{Remaining} = 80 \times \left(\frac{1}{2}\right)^3 = 80 \times \frac{1}{8} = 10 \text{ grams}$$

Example 5: Sound Intensity (Science/Decibels)

Every 10 dB increase in sound intensity is 10 times more powerful. If a sound is 30 dB more intense than a baseline, how many times more powerful is it?

Solution

$$10^{\frac{30}{10}} = 10^3 = 1000 \text{ times}$$

CONCEPT OF LOGARITHMS

Just as subtraction can be used to reverse addition and division can be used to reverse multiplication, there is a concept that can be used to reverse indices. This concept is called logarithm. Thus, indices and logarithms are closely related. The concept of logarithms is an important tool in many fields. For example, banks use logarithms to calculate the compound interest of their customers; biologists use it to determine the rate at which the population of organisms like bacteria grow in a given medium; chemists use it to find the pH of substances and geologists use it to measure the magnitude of an earthquake.

Here you will learn the rules which govern the concept of logarithms.

Laws of Logarithms

When you see $\log_a y$, a is the base and y is the augment. This is read as logarithm of y to the base a . In short, you can say, logy base a . Logarithms to base 10 are known as common logarithms. For common logarithms, sometimes the base is omitted. Thus, $\log y = \log_{10} y$. Common logarithms were very important in computations before the calculator was invented.

1. Since logarithms reverse indices, $a^x = y \iff \log_a y = x$.
2. $\log_a (x \times y) = \log_a x + \log_a y$. You can use this law, known as the addition-product law, if, and only if, the augment is a product.

3. $\log_a(x \div y) = \log_a x - \log_a y$. You can apply this law, known as the subtraction – quotient law, if, and only if, the augment is a quotient.
4. $\log_a y^x = x \log_a y$. You can apply this law, known as the exponent law, if, and only if, the augment has an exponent or power.
5. $\log_a a = 1$. If the augment is the same as the base the answer is always 1
 - i. as $a^1 = a$.
6. $\log_a 1 = 0$. No matter the base, the logarithm of one is always zero as $a^0 = 1$
7. $\log_a y = \frac{\log_x y}{\log_x a}$. This law is used to change the base of a logarithm.

Applying the Laws of Logarithms

Apply the laws of logarithms to solve these problems.

Example 1.14

Given that $\log_2 3 = 1.58$ and $\log_2 5 = 2.32$, find the value of:

1. $\log_2 15$
2. $\log_2 45$
3. $\log_2 \left(\frac{5}{3}\right)$.

Solution

1. $\log_2 15 = \log_2 (3 \times 5) = \log_2 3 + \log_2 5$
 $= 1.58 + 2.32$
 $= 3.90$
2. $\log_2 45 = \log_2 (9 \times 5) = \log_2 9 + \log_2 5 = \log_2 3^2 + \log_2 5$
 $= 2 \log_2 3 + \log_2 5$
 $= 2(1.58) + 2.32$
 $= 3.16 + 2.32$
 $= 5.48$
3. $\log_2 \left(\frac{5}{3}\right) = \log_2 5 - \log_2 3$
 $= 2.32 - 1.58$
 $= 0.74$

Example 1.15

Solve for x : $\log_2 (x - 3) = 4$

Solution

$$\log_2(x - 3) = 4$$

$$\implies x - 3 = 2^4$$

$$\implies x - 3 = 16$$

$$\implies x = 16 + 3 = 19$$

Example 1.16

Simplify: $\log_2 8 + \log_2 4$

Solution

$$\log_2 8 + \log_2 4$$

$$= \log_2 2^3 + \log_2 2^2$$

$$= 3\log_2 2 + 2\log_2 2$$

$$= 3(1) + 2(1)$$

$$= 3 + 2$$

$$= 5$$

$$\text{Alternatively, } \log_2 8 + \log_2 4 = \log_2 (8 \times 4)$$

$$= \log_2 32 = \log_2 2^5 = 5 \log_2 2$$

$$= 5 \times 1 = 5.$$

Example 1.17

Find the value of x given that $\log_3 x + 4\log_x 3 = 5$.

Solution

$$\log_3 x + 4\log_x 3 = 5$$

$$\implies \log_3 x + \log_x 3^4 = 5$$

$$\implies \log_3 x + \log_x 81 = 5$$

$$\implies \log_3 x + \frac{\log_3 81}{\log_3 x} = 5$$

$$\implies \log_3 x + \frac{\log_3 3^4}{\log_3 x} = 5$$

$$\implies \log_3 x + \frac{4 \log_3 3}{\log_3 x} = 5$$

$$\Rightarrow \log_3 x + \frac{4}{\log_3 x} = 5$$

At this point, let $\log_3 x = y$.

$$\Rightarrow y + \frac{4}{y} = 5$$

$$\Rightarrow y(y) + \frac{4}{y} \times y = 5y$$

$$\Rightarrow y^2 + 4 = 5y$$

$$\Rightarrow y^2 - 5y + 4 = 0$$

$$\Rightarrow y^2 - y - 4y + 4 = 0$$

$$\Rightarrow y(y - 1) - 4(y - 1) = 0$$

$$\Rightarrow (y - 1)(y - 4) = 0$$

$$\Rightarrow y - 1 = 0 \text{ or } y - 4 = 0$$

$$\therefore y = 1 \text{ or } y = 4$$

$$\Rightarrow \log_3 x = 1 \text{ or } \log_3 x = 4$$

$$\Rightarrow x = 3^1 \text{ or } x = 3^4$$

$$\Rightarrow x = 3 \text{ or } x = 81.$$

Time for a quick recap of some of the key concepts of indices and logarithms.

Indices or Exponents or Powers:

1. These are mathematical operations that involve raising a number to a specific power known as the index/exponent. This power or exponent or index represents the number of times a base is multiplied by itself.
2. For example, the expression a^b , where ' a ' is the base and ' b ' is the exponent, means we multiply a by itself b number of times.

Logarithms:

1. Logarithms are the inverse operation of indices.
2. The logarithm of a number with respect to a given base tells us what exponent we need to raise the base to in order to obtain that number. For example, if $b^x = a$, then the logarithm is expressed as $\log_b a = x$ and is read as "the logarithm of a to the base b is x " meaning the logarithm of a to base b is the exponent (x) to which we raise the base (b) to get the number (a).

Let us consider some key relationships between indices and logarithms.

1. Logarithms are the inverse of indices.
2. The laws of logarithms are derived from the laws of indices.

3. The logarithmic function “cancels out” or reverses the effect of exponents in expressions or equations. It allows us to find the unknown exponent when we know the base and the result.
4. Logarithms are useful in solving exponential equations, analysing growth rates and working with data that follow exponential patterns.
5. Indices and logarithms are two sides of the same mathematical coin. They complement each other and play essential roles in various fields including science, engineering and finance.

Let us explore areas in which indices and logarithms are applied.

In Banking and Finance

Indices can be used:

1. to calculate interest rates on investments or loans.
For example, the formula for Compound Interest, $A = P \left(1 + \frac{r}{n}\right)^{nt}$
2. in inflation rates to determine the change in prices of items.
3. in analysing stock market prices.

Logarithms can be used:

1. to determine the time it takes an investment to reach a specific value/amount.
2. in assessing financial risk and uncertainty.

In computer science and IT

1. Logarithmic procedures can be used to compress data.
2. Indices are used to calculate compression ratios.
3. Logarithms are used in calculations to secure online transactions. E.g., SSL
4. Google’s search procedures use logarithmic scaling to rank web pages.

Science and Engineering

1. In Physics, indices are used to calculate forces, energies, and velocities
 - a. (e.g., kinetic energy = $\frac{1}{2}mv^2$)
2. Sound intensity is measured in decibels(dB) using logarithmic scales.
3. In Chemistry, pH levels of acids and bases are measured using logarithms.
4. Logarithms are used in signal processing in the compression/decompression in audio and image processing.
5. In epidemic modelling, indices are used to calculate the spread and growth rates of diseases (e.g., exponential growth)
6. The decay of radioactive substances follows an exponential pattern.

7. Half-life (the time for half of a substance to decay) is determined using logarithms.

In our everyday life

1. Logarithmic scales are used to measure sound frequencies in music.
2. In photography, logarithmic exposure compensation adjusts image brightness.

This shows how useful and applicable indices and logarithms are. Explore the internet to look for other areas where they can be applied.

Applications of Common Logarithms

Before calculators were invented, common logarithms were used to make computations because it is easy to work in base 10.

Let us consider some questions and see how to solve them using the concept of common logarithms.

Example 1.18: Compound interest

Mrs. Agbenyegah invested GH¢15 000.00 at 12% interest monthly, how much will she have after 3years? Use the formula: $Amount(A) = P \left(1 + \frac{r}{n}\right)^{nt}$

Solution

$$Amount(A) = P \left(1 + \frac{r}{n}\right)^{nt}$$

where $P = \text{GH¢}15\,000$, $r = 0.12$, $n = 12$, $t = 3$

$$A = 15\,000 \left(1 + \frac{0.12}{12}\right)^{12 \times 3}$$

$$A = 15\,000(1.01)^{36}$$

$$A = 15,000(1.430768784)$$

$$A \approx \text{GH¢} 21461.53$$

Therefore, Mrs. Agbenyegah will have a total amount of GH¢ 21461.53 after 3years.

Example 1.19: Sound (decibels)

A sound wave has an intensity of 0.01 watts/m². What is its decibel level?

Use the formula: Decibel (dB) level = $10 \times \log_{10}\left(\frac{I}{I_0}\right)$

Where I is the intensity of sound and I_0 is the reference intensity, typically 10^{-12} W/m

Solution

$$\text{Decibel Level (L)} = 10 \times \log_{10}\left(\frac{I}{I_0}\right)$$

$$\begin{aligned} L &= 10 \log\left(\frac{0.01}{10^{-12}}\right) \\ &= 10(\log_{10} 0.01 - \log_{10} 10^{-12}) \\ &= 10(\log_{10} 10^{-2} - \log_{10} 10^{-12}) \\ &= 10(-2 \log_{10} 10 - (-12) \log_{10} 10) \\ &= 10(-2 + 12) = 10(10) \\ &= 100 \text{ decibels (dB)} \end{aligned}$$

Therefore, the decibel level of the sound wave is **100dB**.

Example 1.20: Sound

A data compression algorithm reduces a file size from 100MB to 10MB.

What is the compression ratio?

Use the formula: $\text{Common Ratio} = \log\left(\frac{\text{Original Size}}{\text{Compressed Size}}\right)$

Solution

$$\begin{aligned} \text{Compression ratio} &= \log\left(\frac{100}{10}\right) = \log(100) - \log(10) \\ &= \log_{10} 10^2 - \log_{10} 10 \\ &= 2 \log_{10} 10 - \log_{10} 10 \\ &= \log_{10} 10(2 - 1) \\ &\approx 1 \end{aligned}$$

$$\begin{aligned} \text{Alternately, Compression ratio} &= \log\left(\frac{100}{10}\right) = \log(100) - \log(10) \\ &= \log_{10} 10^2 - \log_{10} 10 \\ &= 2 \log_{10} 10 - \log_{10} 10 \text{ since } \log_{10} 10 = 1 \\ &= 2(1) - 1 \\ &\approx 1 \end{aligned}$$

Or, another alternative, $\text{Compression ratio} = \log\left(\frac{100}{10}\right) = \log(10) = \log_{10} 10 = 1$

Therefore, the compression ratio is approximately 1

Example 1.21: Biology and Medicine

In a particular country, a population of bacteria grows from 100 to 1000 cells in 3 hours. What is the growth rate of the bacteria?

Use the formula: $r = \frac{\log\left(\frac{P(t)}{P_0}\right)}{t}$, where:

r is the growth rate (as a decimal)

$P(t)$ is the population at time t

P_0 is the initial population

t is time (in hours)

Solution

$$\text{Growth rate } (r) = \frac{\log\left(\frac{P(t)}{P_0}\right)}{t}$$

$$r = \frac{\log\left(\frac{1000}{100}\right)}{3}$$

$$r = \frac{\log 10}{3} = \frac{1}{3} \approx 0.3333$$

Therefore, the growth rate (r) is approximately 0.3333 per hour.

MODULAR ARITHMETIC

In primary school, you were taught sorting and the divisibility rules of integers. Hopefully you can recall that the divisibility rule of integers is a method used to determine whether one integer is divisible by another without performing the full division.

Some of these rules are;

- 1. Divisibility by 2:** A number is divisible by 2 if its last digit is even (0, 2, 4, 6, or 8).
- 2. Divisibility by 3:** A number is divisible by 3 if the sum of its digits is divisible by 3. For example (0, 3, 21, 63, 84, 102, 258, 1095, 8631, etc.)
- 3. Divisibility by 5:** A number is divisible by 5 if its last digit is 0 or 5.
- 4. Divisibility by 10:** A number is divisible by 10 if its last digit is 0.
- 5. Divisibility by 4:** A number is divisible by 4 if the number formed by its last two digits is divisible by 4. E.g. (112, 224, 1092, 9908, etc.)
- 6. Divisibility by 9:** A number is divisible by 9 if the sum of its digits is divisible by 9. For example (9, 27, 72, 81, 117, 333, 225, 3141, 4203, 7110, 8001, 9032031, etc.)

Divisibility rules work perfectly because one integer divides another without a remainder and they are a quick way to determine whether an integer is divisible by another. Do you know any other divisibility rules? Discuss them with a classmate.

But what happens when one integer is **not** divisible by another? This results in a **remainder**.

For example, dividing 8 by 3. By the divisibility rule, a number can only be divisible by 3 if the sum of its digits is 3. Therefore, we know that this is not divisible and there will be remainder.

$\frac{8}{3} = 2\frac{2}{3}$. This means that 8 divides 3, 2 times with a remainder of 2.

This concept is closely related to **modular arithmetic**, which deals with the remainder when one integer is divided by another.

Introduction and Basic Operations with Modular Arithmetic

With the help of clocks/watches (analogue and digital), perform this activity.

Activity 1.7: Clock arithmetic

For this activity you need both an analogue clock/watch and a digital clock/watch.



Figure 1.4: A clock

Step 1: Identify the various sections or digits on the surface of the clock/watch.

Step 2: Set the time to 12:00 or 00:00 on the analogue clock and 00:00 on the digital clock/watch.

Step 3: Turn the clock forward by 13 hours.

Step 4: Identify the numbers obtained on both the analogue and digital clock/watch and write them down.

Step 5: Repeat steps 2 to 4 using other number of hours.

Step 6: Discuss your findings/ observations with a classmate.

You will have observed that +13hrs is 1:00 on the analogue clock but 13:00 on the digital clock.

If you went forward by 21hrs from the starting point you have 9:00 on the analogue clock but 21:00 on the digital clock. This is because, the numbers wrap round a fixed number, 12:00 (the starting point), on the analogue clock.

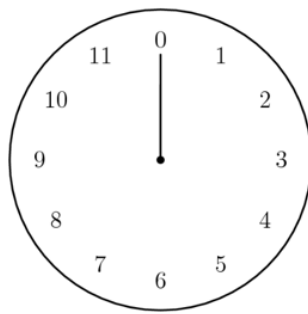


Figure 1.5: A clock

In modular arithmetic, numbers “wrap around” upon reaching a given fixed quantity (this given quantity is known as the modulus) to leave a remainder. Other examples are the days of the week, calendar months, etc.

Modular arithmetic, often referred to as remainder arithmetic, is a mathematical system where an integer is expressed by its remainder after being divided by another integer.

The modulo of any integer is determined by dividing that integer by a specified modulus and identifying the remainder as the result. The term “**mod**” is used as a notation for modulo. That is, “ $a \bmod b$ ” represents the remainder when a is divided by b .

For example, $31(\bmod 7)$ signifies the remainder obtained when 31 is divided by 7.
 $31(\bmod 7) = 31 - 7 = 24 - 7 = 17 - 7 = 10 - 7 = 3$

$9(\bmod 4)$ signifies the remainder obtained when 9 divided by 4. Thus,

$9(\bmod 4) = 9 - 4 = 5 - 4 = 1$. Therefore, $9(\bmod 4) = 1$ and can also be written as:
 $9 = 1(\bmod 4)$

The remainder can be obtained either by:

1. repeatedly subtracting the modulo number from the given number until subtracting any further will result in negative numbers.

For example, $7(\text{mod } 4) = 7 - 4 = 3$. If we try and subtract 4 again, we will have a negative number $\therefore 7(\text{mod } 4) = 3$ or $7 = 3(\text{mod } 4)$

2. dividing the given number by the modulo number and finding the remainder.

For example, if $\frac{d}{c} = A\frac{b}{c}$, where A is the integer part, c is the divisor and b is the remainder, so the value of b is the answer.

For example, $7(\text{mod } 4) = \frac{7}{4} = 1\frac{3}{4}$. The 3 is the remainder.

$$\implies 7(\text{mod } 4) = 3$$

Examples 1.22

1. $7(\text{mod } 4) = 3$ (remainder after dividing 7 by 4)
2. $2(\text{mod } 10) = 2$ (remainder after dividing 2 by 10)
3. $9(\text{mod } 3) = 0$ (remainder after dividing 9 by 3)
4. $81(\text{mod } 5) = 1$ (remainder after dividing 81 by 5)

Hints in finding the modulo:

1. If the dividend is less than the divisor (modulo number), then the dividend remains the answer. For example:
 - a. $3(\text{mod } 7) = 3$
 - b. $1(\text{mod } 5) = 1$
2. Modulo 5 of all numbers ending with any of these numbers; 0, 1, 2, 3, 4 is the last or ending digit. For example:
 - a. $1(\text{mod } 5) = 1$
 - b. $12(\text{mod } 5) = 2$
 - c. $170(\text{mod } 5) = 0$
 - d. $1094(\text{mod } 5) = 4$ etc
3. Modulo 5 of all numbers ending with any of these numbers; 5, 6, 7, 8, 9 is found by subtracting 5 from the last digit. For example:
 - a. $19(\text{mod } 5) = 9 - 5 = 4$. Therefore, $19(\text{mod } 5) = 4$
 - b. $36(\text{mod } 5) = 6 - 5 = 1$

Integers for a given Modulo

Let's take for example, the division of any integer by 3:

$\frac{1}{3}$ has a remainder of 1

$\frac{2}{3}$ has a remainder of 2

$\frac{3}{3}$ has a remainder of 0

$\frac{4}{3}$ has a remainder of 1

It is seen that the possible remainders when any integer divided by 3 are 0, 1 and 2. The divisor (3), is called the modulus and the arithmetic is said to be in modulo 3. Modulo 3 can therefore take one of the values $\{0, 1, 2\}$. Likewise, modulo 5 can take one of the values $\{0, 1, 2, 3, 4\}$,

Modulo 6 can take one of the values $\{0, 1, 2, 3, 4, 5\}$

Modulo 9 can take one of the values $\{0, 1, 2, \dots, 8\}$

In general, modulo n can take one of the values $\{0, \dots, n - 1\}$

You can try this with several other integers and discuss your observations with your classmates and teacher.

Modulo of Negative Numbers

The modulo of any negative number is determined by adding the modulo to the number successively until a positive number, which is the answer, is reached. This implies that we repeatedly add the modulus to the negative number until it gives an answer which is positive.

For example, $-5 \pmod{4} = -5 + 4 = -1 + 4 = 3$

Since 3 is the first positive number, we end the process that is the answer.

Therefore, $-5 \pmod{4} = 3$

Example 1.23

Simplify $-8 \pmod{3}$

Solution

$$\begin{aligned} -8 \pmod{3} &= -8 + 3 = -5 \\ &= -5 + 3 = -2 \\ &= -2 + 3 = 1 \end{aligned}$$

$$\therefore -8 \pmod{3} = 1$$

Example 1.24

What is the value of -33 in modulo 11?

Solution

$$\begin{aligned} -33 \pmod{11} &= -33 + 11 = -22 \\ &= -22 + 11 = -11 \\ &= -11 + 11 = 0 \end{aligned}$$

$$\therefore -33 \pmod{11} = 0$$

Equivalent or Congruent Modulo

In modular arithmetic, two integers **a** and **b** are said to be **equal modulo n**, written as $a \equiv b \pmod{n}$ if they leave the same remainder under the same mod **n**. In other words, a and b are equivalent if their difference is a multiple of **n**.

For example:

1. $30 \pmod{7} = 2$ and $23 \pmod{7} = 2$. Therefore, 30 and 23 are said to be equivalent and written as: $30 \pmod{7} \equiv 23 \pmod{7} = 2$ or $30 \equiv 23 \pmod{7}$
2. If $16 \pmod{5} = 1$ and $91 \pmod{5} = 1$
Then $16 \equiv 91 \pmod{5}$

Basic Operations with Modulo

The basic operations are addition (+), subtraction (−), multiplication (×) and division (÷). The act of performing these operations is known as simplification of modular arithmetic.

This simplification of modular arithmetic is done by;

1. performing the operation first. (+, −, ×, ÷)
2. converting the number obtained to the given modulo.

Addition

The sum of two or more numbers in a given modulo is found by adding the numbers first before converting to the given modulo.

Example 1.25

1. $4 + 5 \pmod{5} = 9 \pmod{5} = 4$

2. $8 + 12 \pmod{12} = 20 \pmod{12} = 8$
3. $-3 + 31 + 4 \pmod{9} = 32 \pmod{9} = 5$
4. $(5 \pmod{3}) + (2 \pmod{3}) = (5 + 2) \pmod{3} = 7 \pmod{3} = 1$

Subtraction

The difference of two or more numbers in a given modulo is found by first finding the difference between the numbers before converting to the given modulo.

Example 1.26

1. $99 \pmod{5} - 34 \pmod{5} = (99 - 34) \pmod{5} = 65 \pmod{5} = 0$
2. $(7 \pmod{4}) - (2 \pmod{4}) = (7 - 2) \pmod{4} = 5 \pmod{4} = 1$
3. $12 - 14 \pmod{6} = -2 \pmod{6} = 4$

Multiplication

The product of two or more numbers in a given modulo is obtained by multiplying the numbers first before converting to the given modulo.

Example 1.27

1. $(3 \pmod{5}) \times (2 \pmod{5}) = (3 \times 2) \pmod{5} = 6 \pmod{5} = 1$
2. $12 \times 12 \pmod{4} = 144 \pmod{4} = 0$
3. $6 \times 4 \pmod{7} = 24 \pmod{7} = 3$

Let us investigate the following properties of operation involving modulo arithmetic.

Properties of Modular Arithmetic

Commutative Property

Commutativity in operations involving modulo (e.g., addition, multiplication under mod)

Addition and multiplication modulo a number are commutative:

- $(a + b) \pmod{n} = (b + a) \pmod{n}$
- $(a \times b) \pmod{n} = (b \times a) \pmod{n}$

So commutativity applies to modular addition and multiplication.

Note!

But the statement:

“ $a \bmod b \neq b \bmod a$ ”

is **not** about commutativity. That’s just comparing two different operations.

Let’s look at an example:

- $7 \bmod 4 = 3$
- $4 \bmod 7 = 4$

Clearly,

$$7 \bmod 4 \neq 4 \bmod 7$$

So, **modulo itself (i.e., $a \bmod b$) is not commutative.**

Associative Property

This property is applicable to both addition and multiplication in modulo arithmetic. That is:

$$(a + b) + c \bmod n = a + (b + c) \bmod n$$

$$(a \times b) \times c \bmod n = a \times (b \times c) \bmod n$$

Let us verify:

$$\text{For example: } (2 + 3) + 4 \bmod 5 = 2 + (3 + 4) \bmod 5$$

Solving the Left-Hand Side (LHS) and Right- Hand Side (RHS) concurrently,

$$5 + 4 \bmod 5 = 2 + 7 \bmod 5$$

$$9 \bmod 5 = 9 \bmod 5$$

$$4 = 4$$

Since $(LHS) = (RHS) = 4$, the additive property holds.

Similarly

$$(2 \times 3) \times 4 \bmod 5 = 2 \times (3 \times 4) \bmod 5$$

$$6 \times 4 \bmod 5 = 2 \times 12 \bmod 5$$

$$24 \bmod 5 = 24 \bmod 5$$

$$4 = 4$$

Since $(LHS) = (RHS) = 4$, the multiplicative property also holds.

Therefore, associative property holds for modular arithmetic.

Identity

This refers to the existence of an additive and multiplicative identity in modular arithmetic such that, when combined with any number under a specific operation, leaves that number unchanged.

1. The additive identity in modular arithmetic is **0**

That is, $a + 0 \pmod n = 0 + a \pmod n = a \pmod n$

For example: $3 + 0 \pmod 2 = 0 + 3 \pmod 2 = 3 \pmod 2 = 1$

2. The multiplicative identity in modular arithmetic is **1**

That is, $a \cdot 1 \pmod n = 1 \cdot a \pmod n = a \pmod n$

For example: $5 \times 1 \pmod 3 = 1 \times 5 \pmod 3 = 5 \pmod 3 = 2$

Inverse

This refers to the existence of additive and multiplicative inverses in modular arithmetic such that, when combined with any number under a specific operation, yields the identity element.

1. The **additive inverse** of modular arithmetic is the value that, when added to a number, gives a result of $0 \pmod n$. That is, $a + b = 0 \pmod n$, This means that **b** is the value that, when added to **a**, results in the identity element 0.

Example 1.28

a) $3 + 1 \pmod 4 = 0 \pmod 4 = 0$, 1 is the additive inverse of 3 in $\pmod 4$

b) $5 + 2 \pmod 7 = 7 \pmod 7 = 0$, 2 is the additive inverse of 5 in $\pmod 7$

2. The **multiplicative inverse** is the value that, when multiplied by a number, gives a result of $1 \pmod n$. That is $a \cdot b = 1 \pmod n$

This means that **b** is the value that, when multiplied by **a**, gives a result that is congruent to 1 modulo n

Example 1.29

a) $3 \times 5 = 1 \pmod 7$, therefore, 5 is the multiplicative inverse of 3 in $\pmod 7$

b) $4 \times 3 = 1 \pmod 11$, so 3 is the multiplicative inverse of 4 in $\pmod 11$

Explore the internet for more examples, solve them and discuss them with your classmates. How can we find the modulo number if a number and its remainder are given or known but the modulo number is unknown? For example, $24 \pmod x = 3$

Let us go through the activity below to determine the missing value.

Activity 1.8: Finding the modulo number in a given equation

Determine the value of x in $24 \pmod{x} = 3$

Step 1: Find the difference between the numbers

Thus, $24 - 3 = 21$

Step 2: find all the factors of the value obtained

Factors of $21 = \{1, 3, 7, 21\}$

Step 3: Substitute each of the factors in the given equation.

when $x = 1$, $24 \pmod{1} = 0$, $x \neq 1$

when $x = 3$, $24 \pmod{3} = 0$, $x \neq 3$

when $x = 7$, $24 \pmod{7} = 3$, $x = 7$

when $x = 21$, $24 \pmod{21} = 3$, $x = 21$

Step 4: identify the ones that make the statement true (ie, satisfy the statement), as the value of the variable.

$x = 7$ or 21 because 7 and 21 both satisfy the equation.

You can search for other methods on the internet.

Now, let's try another example.

Example 1.30

Find p , if $52 = 3 \pmod{p}$ where $0 \leq p \leq 10$

Solution

$52 = 3 \pmod{p}$

$52 - 3 = 49$

Factors of $49 = \{1, 7, 49\}$

when $p = 1$, $52 \pmod{1} = 0$, $p \neq 1$

when $p = 7$, $52 \pmod{7} = 3$, $p = 7$

when $p = 49$, $52 \pmod{49} = 3$, $p = 49$

7 and 49 both satisfy the equation. However, the question states that $0 \leq p \leq 10$,

Therefore, $p = 7$

Example 1.31

You have 36 cookies and want to share them equally among your friends. If you have a remainder of 4 cookies after sharing, how many friends do you have?

Solution

Let n represent number of friends

$$\implies 36 = 4 \pmod{n}$$

$$36 - 4 = 32$$

Factors of 32 = $\{1, 2, 4, 8, 16, 32\}$

When $n = 1$, $36 \pmod{1} = 0$, $n \neq 1$

When $n = 2$, $36 \pmod{2} = 0$, $n \neq 2$

When $n = 4$, $36 \pmod{4} = 0$, $n \neq 4$

When $n = 8$, $36 \pmod{8} = 4$, $n = 8$

When $n = 16$, $36 \pmod{16} = 4$, $n = 16$

When $n = 32$, $36 \pmod{32} = 4$, $n = 32$

$n = 8$ or 16 or 32

Therefore, the number of friends is either 8 or 16 or 32

Finding the Number (unknown value) Under a Given Modulo

How do we determine the number which is being operated on by a given modulo? For example, $7x = 1 \pmod{2}$. Work through this activity to help you find the value of x .

Activity 1.9: Finding the unknown value in a given equation

Find the possible value(s) of x such that $7x = 1 \pmod{2}$

Step 1: List all the possible remainders of the given modulo. They are $\{0, 1\}$

Step 2: Substitute each of the remainder values into the given equation.

when $x = 0$, $7(0) \pmod{2} = 0 \pmod{2} = 0$, $x \neq 0$

when $x = 1$, $7(1) \pmod{2} = 7 \pmod{2} = 1$, $x = 1$

Step 3: Identify the value(s) of the variable (x) that makes the statement true.

Thus $x = 1$ makes the statement true.

Example 1.32

Find the value(s) of x in the given equation $5x = 2(mod\ 4)$.

Solution

$$5x = 2(mod\ 4) \text{ or } 5x(mod\ 4) = 2$$

In $mod(4)$, x can take on values from 0 to $n - 1$, that is $\{0,1,2,3\}$

Substituting:

$$\text{when } x = 0, 5(0)mod\ 4 = 0(mod\ 4) = 0, x \neq 0$$

$$\text{when } x = 1, 5(1)mod\ 4 = 5(mod\ 4) = 1, x \neq 1$$

$$\text{when } x = 2, 5(2)mod\ 4 = 10(mod\ 4) = 2, x = 2$$

$$\text{when } x = 3, 5(3)mod\ 4 = 15(mod\ 4) = 3, x \neq 3$$

$$\therefore x = 2$$

Example 1.33

Given that $2y + 1 = 3(mod\ 7)$, what is the value of y ?

Solution

$$2y + 1 = 3(mod\ 7)$$

In $mod(7)$, y can take on values of $\{0,1, 2, 3, 4, 5, 6\}$

By substitution, let's investigate:

$$\text{When } y = 0, 2(0) + 1(mod\ 7) = 1(mod\ 7) = 1, y \neq 0$$

$$\text{When } y = 1, 2(1) + 1(mod\ 7) = 3(mod\ 7) = 3, y = 1$$

$$\text{When } y = 2, 2(2) + 1(mod\ 7) = 5(mod\ 7) = 5, y \neq 2$$

$$\text{When } y = 3, 2(3) + 1(mod\ 7) = 7(mod\ 7) = 0, y \neq 3$$

$$\text{When } y = 4, 2(4) + 1(mod\ 7) = 9(mod\ 7) = 2, y \neq 4$$

$$\text{When } y = 5, 2(5) + 1(mod\ 7) = 11(mod\ 7) = 4, y \neq 5$$

$$\text{When } y = 6, 2(6) + 1(mod\ 7) = 13(mod\ 7) = 6, y \neq 6$$

$$\therefore y = 1$$

Search for other methods that can be used to solve the above examples and look for more examples from the extended reading materials provided below or from the internet.

Modular arithmetic is applicable in several areas and even in our everyday lives.

Let us consider some of these areas;

Applications of Modular Arithmetic

1. In our daily lives, modular arithmetic can be used to determine;
 - i. Time: Modular arithmetic can be used to determine what time it is after a number of hours. The 12-hour clock system is essentially modular arithmetic using modulo 12 (for example, 10 hours after 3pm is 1 am).
 - ii. Market days in our communities: Market days in our communities rotate in a cyclic manner. For instance, in Ho, market days are every five days counting from the recent market day. Since there are only 7 days in a week, the market days wrap around it.

The figure below shows how cyclic the days of the week are.

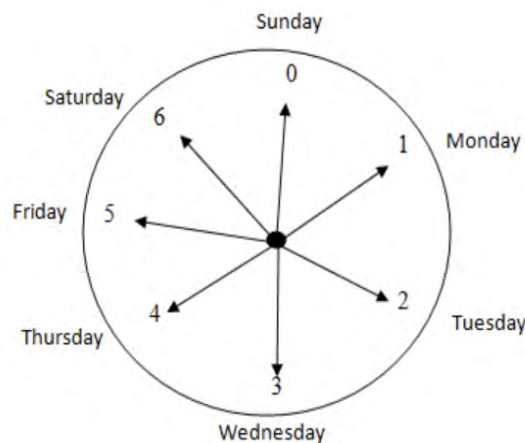


Figure 1.6: Days of the week clock

For example:

- a) If the recent market day is Saturday, then the next market day will be Wednesday.
- b) If today is Thursday (represented by the number 4), what day will it be in 11 days?

Solution

$$\text{Thursday (4)} + 11 \text{ days} = 15$$

$$\Rightarrow 15(\text{mod } 7) = 1,$$

From the cycle, 1 represents Monday. Therefore, 11 days after Thursday is Monday.

- c) The calendar is also designed using the concept of modular arithmetic (i.e. days of the week, months and year)
- 2. **Finding the remainder:** for example, $27 \bmod 4 = 3$ (i.e. the remainder obtained when 27 is divided by 4)
- 3. **In Computer Science:** It is widely applied in various computer science areas like pseudo random number generation, finite field arithmetic, and generating permutations.
- 4. **Error Detection and Correction:** It's used in checksums (adding digits and using the remainder for error detection) and cyclic redundancy checks (CRC) for data transmission reliability.

There are other areas you can look out for. Do search for them and discuss them with your classmates.

EXTENDED READING

1. *Aki – Ola Series: Core Mathematics for Senior High Schools in West Africa*, Millennium edition 5 (Pages 33– 40)
2. Baffour A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 398 - 408)

REVIEW QUESTIONS 1

1. Which of the following is not a perfect square?
 - A. 225
 - B. 625
 - C. 652
 - D. 841
2. Simplify the following surds, if x is natural number:
 - a. $\sqrt{24}$
 - b. $\sqrt{45}$
 - c. $\sqrt{2x^2}$
 - d. $\sqrt{4x}$.
3. The dimensions of a room are such that its length is $\sqrt{108}$ m longer than the width. If the width of the room is 3m, find the length of the room as a surd in its simplest form.
4. Working alone or with a friend, rationalise the following:
 - a. $\frac{6}{\sqrt{3}}$
 - b. $\frac{2 - \sqrt{3}}{2\sqrt{6}}$
 - c. $\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}$
5. Simplify:
 - a. $\sqrt{24} + \sqrt{54}$
 - b. $\sqrt{200} - 3\sqrt{32} + \sqrt{72}$
 - c. $\sqrt{18} - 3\sqrt{50} + \sqrt{27}$
6. Expand and simplify $(\sqrt{5} + \sqrt{3})(2\sqrt{5} - \sqrt{3})$, leaving your answer as a surd in its simplest form.
7. Simplify and leave your answer as a surd in its simplest form.
 - a. $\sqrt{48} - \sqrt{6}$
 - b. $\sqrt{2} - \sqrt{3} \times \sqrt{5} \div \sqrt{6}$.

8. Given that $\sqrt{5} = 2.236$, evaluate, correct to one decimal place, $2\sqrt{5}(4 - 3/\sqrt{5})$.
9. The length of the hypotenuse, AC, of a right-angled triangle ABC is $12\sqrt{5}$ cm. The length of the side AB is $8\sqrt{3}$ cm. Find the length of the side BC, leaving your answer in surd form.
10. Simplify the following:
 - a. $125^{-\frac{1}{3}}$
 - b. $\frac{8^{\frac{1}{3}} \times 4^{\frac{1}{2}}}{32^{\frac{1}{5}} \times 16^{\frac{1}{2}}}$
11. Find the value of x if $2^{x+2} \times 8^x = 1$.
12. Simplify: $\frac{3^5 \times 3^2 \times 3^{-9}}{3^{-7} \times 3^2}$
13. Find the value of y :
 - a. $\log_y 8 = \frac{1}{3}$
 - b. $\log_{27} 3^{2y} = 2$
14. Without the use of four-figure tables or calculators, evaluate:
 - a. $\log_6 36 + \log_6 6\sqrt{6} - \log_6 216$
 - b. $\frac{\log_{36} 6}{\log_{49} 7}$
 - c. $\frac{\log_3 27 - \log_3 3\sqrt{3}}{\log_{3\sqrt{3}} \frac{1}{\sqrt{3}} - \log_3 \sqrt{3}}$
15. Find the remainder when 100 is divided by 7
16. What is the remainder when 123456789 is divided by 9?
17. Find the modular multiplicative inverse of 5 (mod 6).
18. Show that 17 and 29 are congruent under modulo 4.
19. If today is Tuesday (represented by the number 3), what day will it be in 15 days?
20. Akosua can only access her results online 14 hours after it has been uploaded on their school's website. If the results were uploaded at exactly 13:00, at what time will she be able to access it?
21. You have 19 cookies and want to share them equally among your friends. If you have a remainder of 4 cookies after sharing, how many friends do you have?

SECTION

2

EQUATIONS AND INEQUALITIES



ALGEBRAIC REASONING

Applications of Expressions, Equations and Inequalities

INTRODUCTION

In this section, you will explore solving **simultaneous linear equations** in two variables using three key methods: **elimination**, **substitution** and the **graphical method**. You will begin by analysing two linear equations, learning how to eliminate one variable to solve the system algebraically. Substitution, where one equation is rearranged and substituted into the other, will also be practised. You will then solve these equations graphically by plotting and identifying the point where the lines intersect. You will also apply your understanding to **real-life problems**, modelling these scenarios as simultaneous equations and solving them. The section emphasises practical applications, ensuring you can interpret your solutions and relate them to everyday contexts.

KEY IDEAS

- **Elimination method:** Add or subtract equations after manipulating coefficients to eliminate one variable, then solve for the remaining variable.
- **Formulate equations:** Translate the problem's context into two linear equations involving the identified variables.
- **Intersection point:** The point where the two lines intersect. In simultaneous equations this is the solution, if it exists.
- **Linear equation:** is a mathematical equation that shows a straight-line relationship between two variables
- **Simultaneous equation:** is a set of two or more equations that share the same variables and are solved together.
- **Substitution method:** One equation is manipulated to define one variable in terms of the other. This new definition of the variable is then substituted into the second equation and solved.

REVIEW OF PREVIOUS KNOWLEDGE

In year one, you learned how to solve linear equations with one variable. Let's revisit some examples as a quick review. This will lay the groundwork for better understanding of the concept of solving simultaneous equations.

Example 2.1

Solve $4x + 9 = 33$

Solution

Given: $4x + 9 = 33$

Subtract 9 from both sides to isolate the term with x :

$$4x + 9 - 9 = 33 - 9$$

$$4x + 0 = 24$$

$$4x = 24$$

Divide both sides by 4 to solve for x :

$$x = \frac{24}{4}$$

$$x = 6$$

Therefore, the solution is $x = 6$.

Example 2.2

Solve $7y - 12 = 2y + 8$

Solution

Given: $7y - 12 = 2y + 8$

Subtract $2y$ from both sides to get all y terms on one side:

$$7y - 2y - 12 = 2y - 2y + 8$$

$$5y - 12 = 8$$

Add 12 to both sides to isolate the term with y :

$$5y - 12 + 12 = 8 + 12$$

$$5y = 20$$

Divide both sides by 5 to solve for y :

$$y = \frac{20}{5}$$

$$y = 4$$

Therefore, the solution is $y = 4$.

Example 2.3

Solve $\frac{3y}{4} + \frac{1}{8} = 3$

Solution

Given: $\frac{3y}{4} + \frac{1}{8} = 3$

Multiply the entire equation by 8 (LCM of 4 and 8) to remove the fractions:

$$8\left(\frac{3y}{4}\right) + 8\left(\frac{1}{8}\right) = 8 \times 3$$

$$6y + 1 = 24$$

Isolate y by subtracting 1 from both sides:

$$6y + 1 - 1 = 24 - 1$$

$$6y = 23$$

Divide both sides by 6 to solve for y:

$$y = \frac{23}{6}$$

So, the solution is $y = \frac{23}{6}$.

Now that we've completed the revision, let's explore how to solve simultaneous equations using the elimination method.

SOLVING SIMULTANEOUS EQUATIONS USING THE ELIMINATION METHOD

To grasp the elimination method for simultaneous equations, keep the following conditions in mind.

1. When solving simultaneous equations, you are looking for a solution which is true for both equations simultaneously, ie at the same time.
2. When we solve simultaneous linear equations with two unknowns, our goal is to find the value of each unknown that satisfies both equations at the same time.
3. Ensure that the coefficient (the number in front) of one of the unknowns is the same in both equations. This will help us eliminate that unknown.
4. We can eliminate this matching unknown by either adding or subtracting the two equations.

Perform the following activities on how to solve simultaneous equation using the Elimination Method

Activity 2.1: Solving simultaneous equation

Solve $x + y = 5$ and $x - y = 1$ simultaneously.

Solution

Let:

Yellow shapes represent positive numbers.

Red shapes represent negative numbers.

x represent circular shapes.

y represent square shapes.

Constants be represented by triangular shapes.

If the variable is x and has a positive value, represent it with a yellow circle.

If x has a negative value, use a red circle.

For the variable y , use a yellow square for positive values and a red square for negative values.

For constants, represent positive values with a triangular shape and negative values with a red triangular shape

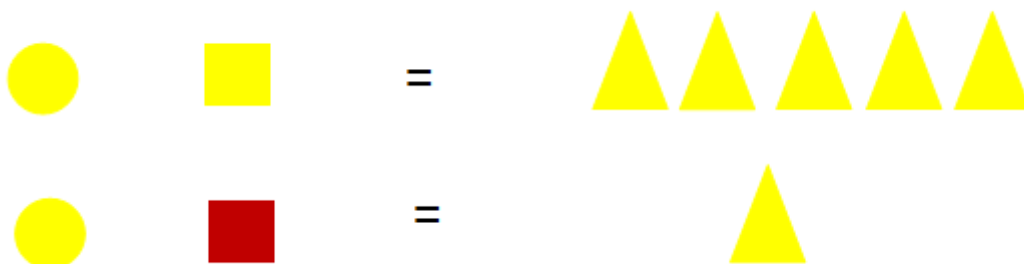
Using the elimination method:

Step 1: Align the equations for clarity

$$x + y = 5 \text{ (Equation 1)}$$

$$x - y = 1 \text{ (Equation 2)}$$

Step 2: Use the specified shapes to represent the different parameters in the given equations: a yellow circle for a positive x , a yellow square for a positive y , a red square for a negative y , 5 yellow triangular shapes for positive constant in equation 1, and a yellow triangular shape for the constant in equation 2.



Step 3: Add the shapes to equations to eliminate y shape



$$2x = 6y$$

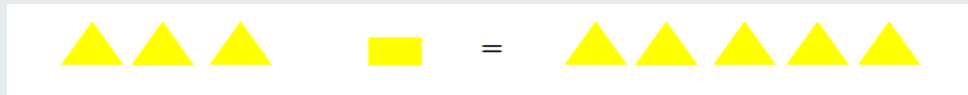
Step 4: You will now share the 6 constants numbers between the two x variables.



$$x = 3y$$

$$x = 3$$

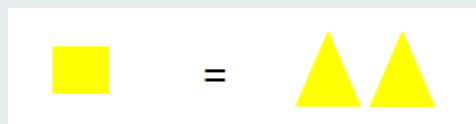
Step 5: Substitute the value of $x = 3$ (yellow circles) into either of the original equations to find y (yellow triangles). Using equation 1.



$$3y + z = 5y$$

Now, you have $3 + y = 5$

Step 6: Now taking one (circle) and one (triangle) from each side until no circles are left at the left hand side will lead to:



$$z = 2y$$

$$y = 2$$

Therefore, the solution is: $x = 3, y = 2$

Confirm that this is correct, but substituting into equation 2:

$$x - y = 1$$

$3 - 2 = 1$ is true, therefore, the solution is correct.

Activity 2.2: Solving simultaneous equation

Solve the simultaneous equations, $2x + 3y = 12$ and $4x - 3y = 6$

Solution

Using the elimination method, follow these steps:

Step 1: Align the equations for clarity

$$2x + 3y = 12 \text{ (Equation 1)}$$

$$4x - 3y = 6 \text{ (Equation 2)}$$

Step 2: Look for a variable with the same magnitude coefficient. If the two coefficients have the same sign subtract the equations, if they are different, add them. In this case, we add the two equations to eliminate y :

$$(2x + 3y) + (4x - 3y) = 12 + 6$$

$$2x + 4x = 18$$

$$6x = 18$$

Step 3: Now, divide both sides by 6 to find x

$$x = \frac{18}{6}$$

$$x = 3$$

Step 4: Substitute the value of $x = 3$ into either of the original equations to find y . We will use Equation 1:

$$2x + 3y = 12$$

$$2(3) + 3y = 12$$

$$6 + 3y = 12$$

Step 5: Isolate y by subtracting 6 from both sides to solve for y

$$6 - 6 + 3y = 12 - 6$$

$$3y = 6$$

Step 6: Divide both sides by 3

$$y = \frac{6}{3}$$

$$y = 2$$

The solution to the simultaneous equations is: $x = 3, y = 2$

To confirm that this is correct, substitute the values we have found into the equation we did not use. In this case, equation 2:

$4(3) - 3(2) = 12 - 6 = 6$, as this is correct, we can be confident in our solution.

Activity 2.3: Solving simultaneous equation

Solve the simultaneous equations, $2x + 3y = 12$ and $4x - y = 5$

Solution

Using the elimination method, follow these steps:

Step 1: Align the equations for clarity

$$2x + 3y = 12 \text{ (Equation 1)}$$

$$4x - y = 5 \text{ (Equation 2)}$$

Step 2: Match the coefficients

To eliminate one variable, you can multiply (equation 2) by 3 to make the coefficients of y in both equations the same magnitude.

(Note, you could also have chosen to multiply (Equation 1) by 2 to make the coefficients of x the same, either one would give you the same answers.)

$$3(4x - y) = 3(5)$$

This gives you:

$$12x - 3y = 15 \text{ (Equation 3)}$$

Now, we have:

$$2x + 3y = 12, \text{ (Equation 1) = The equation we have not touched yet}$$

$$12x - 3y = 15, \text{ (Equation 3) = The manipulated equation 2}$$

Step 3: As the y coefficients have different signs; you will add Equation 1 and Equation 3 to eliminate y :

$$(2x + 3y) + (12x - 3y) = 12 + 15$$

$$14x = 27$$

Step 4: Now, divide both sides by 14 to find x :

$$x = \frac{27}{14}$$

Step 5: Now, substitute $x = \frac{27}{14}$ back into one of the original equations to find y . We will use Equation 1:

$$2\left(\frac{27}{14}\right) + 3y = 12$$

$$\frac{54}{14} + 3y = 12$$

$$\frac{54}{14} - \frac{54}{14} + 3y = 12 - \frac{54}{14}$$

$$3y = 12 - \frac{54}{14}$$

$$3y = \frac{57}{7} \text{ (Divide both sides by 3)}$$

$$y = \left(\frac{57}{7}\right) \div 3$$

$$y = \left(\frac{57}{7}\right) \times \frac{1}{3} = \frac{57}{21}$$

$$y = \frac{19}{7}$$

The solution to the simultaneous equations is: $x = \frac{27}{14}$, $y = \frac{19}{7}$

To confirm that this is correct, substitute the values we have found into the equation we did not use. In this case, equation 2:

$4\left(\frac{27}{14}\right) - \frac{19}{7} = \frac{54}{7} - \frac{19}{7} = \frac{35}{7} = 5$, as this is correct, we can be confident in our solution.

For further practice, visit YouTube and search for tutorials on solving simultaneous equations using the elimination method. Watching these videos will help you better understand the steps and techniques involved. Make sure to practise what you learn.

Now that you have learned how to solve simultaneous equations using the elimination method, you can apply this knowledge to solve simultaneous equations using the substitution method with ease. Let's move on to solving simultaneous linear equations in two variables using the substitution method.

SOLVING SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES USING THE SUBSTITUTION METHOD

The substitution method involves rearranging one of the equations for one variable, making it the subject of the equation, and then substituting that expression into the other equation. This transforms the system of equations into a single equation with one variable, making it easy to solve. Once we have the value for one variable, we can substitute this in to find the other unknown variable.

Follow these steps to solve simultaneous linear equations with two variables using the substitution method:

1. Rearrange one equation to solve and have one variable as the subject.
2. Substitute the expression for that variable into the other equation.
3. Simplify and solve for the remaining variable.
4. Substitute back to find the other variable.
5. Verify your solution by substituting both values back into the original equation. This step is optional, but it is good practice.

Let's carry out the following tasks on solving simultaneous equations using the substitution method:

Activity 2.4: Solving simultaneous equation

Solve the simultaneous equation $2x + y = 7$ and $3x - y = 4$.

Solution

Step 1: Choose an equation to rearrange to solve for one of the variables:

Choosing Equation 1: $2x + y = 7$

Step 2: Solve for one variable (y):

$$y = 7 - 2x$$

Step 3: Substitute the new expression for y into Equation 2:

$$3x - (7 - 2x) = 4$$

Step 4: Simplify and solve for x :

$$3x - 7 + 2x = 4$$

$$5x - 7 = 4$$

$$5x = 11$$

$$x = \frac{11}{5}$$

Step 5: Substitute x back into one of the original equations to find y . Using Equation 1:

$$2\left(\frac{11}{5}\right) + y = 7$$

$$\frac{22}{5} + y = 7$$

Subtract $\frac{22}{5}$ from both sides of the equation to isolate y :

$$\frac{22}{5} - \frac{22}{5} + y = 7 - \frac{22}{5}$$

$$y = \frac{13}{5}$$

The solution to the simultaneous equations is: $x = \frac{11}{5}$, $y = \frac{13}{5}$

Verification

$$\text{Equation 1: } 2\left(\frac{11}{5}\right) + \frac{13}{5} = \frac{22}{5} + \frac{13}{5} = 7 \text{ (True)}$$

$$\text{Equation 2: } 3\left(\frac{11}{5}\right) - \frac{13}{5} = \frac{33}{5} - \frac{13}{5} = 4 \text{ (True)}$$

Activity 2.5: Solving simultaneous equation

Solve the simultaneous equation $2x + 3y = 12$ and $x - y = 1$

Solution

$$2x + 3y = 12 \text{ (Equation 1)}$$

$$x - y = 1 \text{ (Equation 2)}$$

Step 1: Choose an equation to solve for one variable. Choosing equation 2:

$$x - y = 1$$

Step 2: Solve for one variable (x)

$$x = y + 1$$

Step 3: Substitute the new expression for x into equation 1:

$$2(y + 1) + 3y = 12$$

Step 4: Simplify and solve for y :

$$2y + 2 + 3y = 12$$

$$5y + 2 = 12$$

$$5y = 10$$

$$y = 2$$

Step 5: Substitute $y = 2$ back into one of the original equations to find x .
Choosing equation 2:

$$x - 2 = 1$$

$$x = 3$$

Verify by plugging (x, y) into both original equations:

Equation 1: $2(3) + 3(2) = 6 + 6 = 12$ (True)

Equation 2: $3 - 2 = 1$ (True)

SOLVING LINEAR EQUATIONS IN TWO VARIABLES USING THE GRAPHICAL METHOD

The graphical method involves plotting linear equations on a coordinate plane to visually identify the solution to simultaneous equations. Each equation represents a straight line and the point where the lines intersect is the solution to the system of equations.

Follow the steps below to solve simultaneous equation by using graphical method:

Step 1: Write the equations in slope-intercept form:

a. Convert each equation into the form $y = mx + c$, where:

m is the slope (gradient) of the line.

c is the y -intercept (where the line crosses the y -axis).

Step 2: Plot the lines:

- a** For each equation, choose at least two values of x , and calculate the corresponding values of y .
- b** Plot these points on a coordinate plane and draw the line through them, extending the lines across the whole of your graph paper.

Step 3: Find the intersection point:

- a** After plotting both lines, observe where they intersect.
- b** The coordinates of this intersection point represent the solution to the system of equations.

Interpreting Graphs of Linear Equations in Two Variables

1. If two lines intersect at a single point, the system has one unique solution. This point is the common solution for both equations.
2. If two lines are parallel, they never intersect, meaning the system has no solution (inconsistent system).
3. If two lines overlap (i.e., they are the same line), the system has infinitely many solutions, as all points on the line satisfy both equations.

Note that when two lines intersect, it shows where two conditions or relationships balance or agree.

Now that you have a basic understanding of how to solve simultaneous equations graphically, let's engage in the following activities to deepen our grasp of these concepts:

Activity 2.6: Graphing simultaneous equation

Solve $-2x + y = 1$ and $x + y = 4$

Solution

Step 1: Rewrite each equation in slope-intercept form (i.e., $y = mx + c$)

For equation 1: $-2x + y = 1$

Solve for y : $y = 2x + 1$

For equation 2: $x + y = 4$

Solve for y : $y = -x + 4$

Step 2: Create a table of values for each equation.

Choose values of x to calculate corresponding values of y for each equation.

For Equation 1: $y = 2x + 1$

Table 2.1: *x* and *y* coordinates for equ. 1

<i>x</i>	<i>y</i>
0	1
1	3
2	5

For Equation 2: $y = -x + 4$

Table 2.2: *x* and *y* coordinates for equ. 2

<i>x</i>	<i>y</i>
0	4
1	3
2	2

Step 3: Plot the points on a graph

Using the tables from Step 2, plot the points on a graph:

For: $y = 2x + 1$, plot the points (0, 1), (1, 3) and (2, 5).

For: $y = -x + 4$, plot the points (0, 4), (1, 3) and (2, 2).

Step 4: Draw the lines

Draw a straight line through the points for each equation, extending your lines across the whole of the graph paper. The lines represent the two equations as shown below:

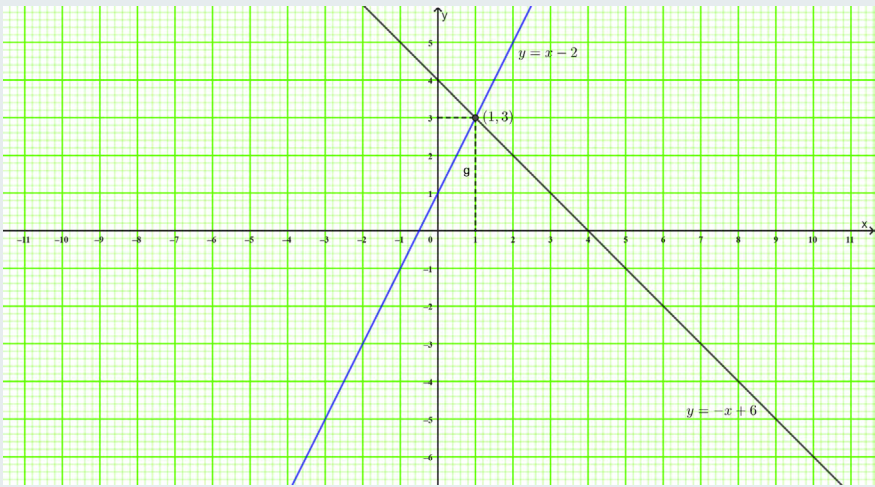


Figure 2.1: Graph of a simultaneous equation

Step 5: Find the point of intersection

The solution to the system of equations is the point where the two lines intersect. In this case, the lines intersect at the point (1, 3).

Step 6: Interpret the solution

The coordinates of the intersection point (1, 3) represent the solution to the system of equations. This means that:

$$x = 1 \text{ and } y = 3$$

These values satisfy both equations simultaneously.

Activity 2.7: Graphing simultaneous equation

Solve the simultaneous equation $2x + y = 6$ and $x - y = 1$ by using the graphical method

Solution:

Step 1: Rewrite each equation in slope-intercept form (i.e., $y = mx + c$)

For equation 1: $2x + y = 6$

Solve for y : $y = -2x + 6$

For equation 2: $x - y = 1$

Solve for y : $y = x - 1$

Step 2: Create a table of values for each equation.

Choose values of x to calculate corresponding values of y for each equation.

For $y = -2x + 6$

Table 2.3: x and y coordinates for equ. 1

x	y
0	6
1	4
2	2

For $y = x - 1$

Table 2.4: x and y coordinates for equ. 2

x	y
0	-1
1	0
2	1

Step 3: Plot the points on a graph

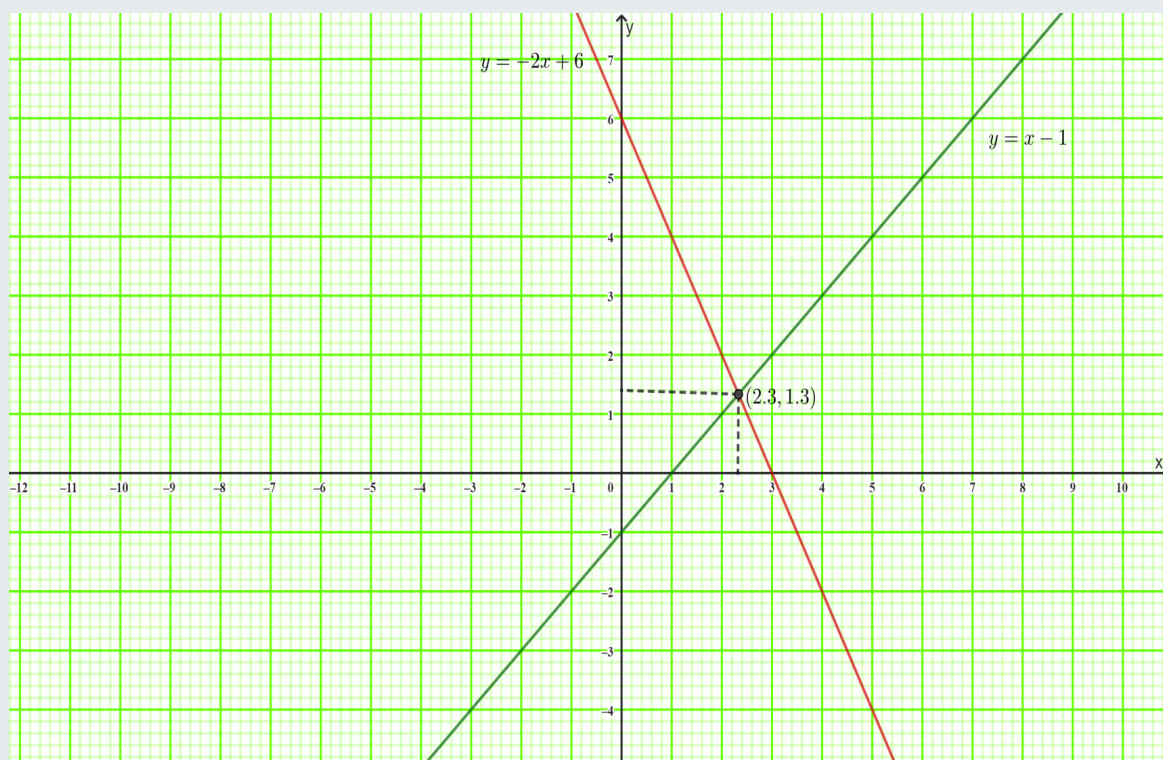
Using the tables from Step 2, plot the points on a graph:

For $y = -2x + 6$, plot the points (0, 6), (1, 4) and (2, 2).

For $y = x - 1$, plot the points (0, -1), (1, 0) and (2, 1).

Step 4: Draw the lines

Draw a straight line through the points for each equation. The lines represent the two equations as shown below:

**Figure 2.2:** Graph of a simultaneous equation**Step 5:** Find the point of intersection

The solution to the system of equations is the point where the two lines intersect. In this case, the lines intersect at the point (2.3, 1.3).

Step 6: Interpret the solution

The coordinates of the intersection point (2.3, 1.3) represent the solution to the system of equations. This means that:

$$x = 2.3 \text{ and } y = 1.3$$

These values satisfy both equations simultaneously.

Activity 2.8: Graphing simultaneous equation

Study the graph below carefully and use it to answer the questions that follow.

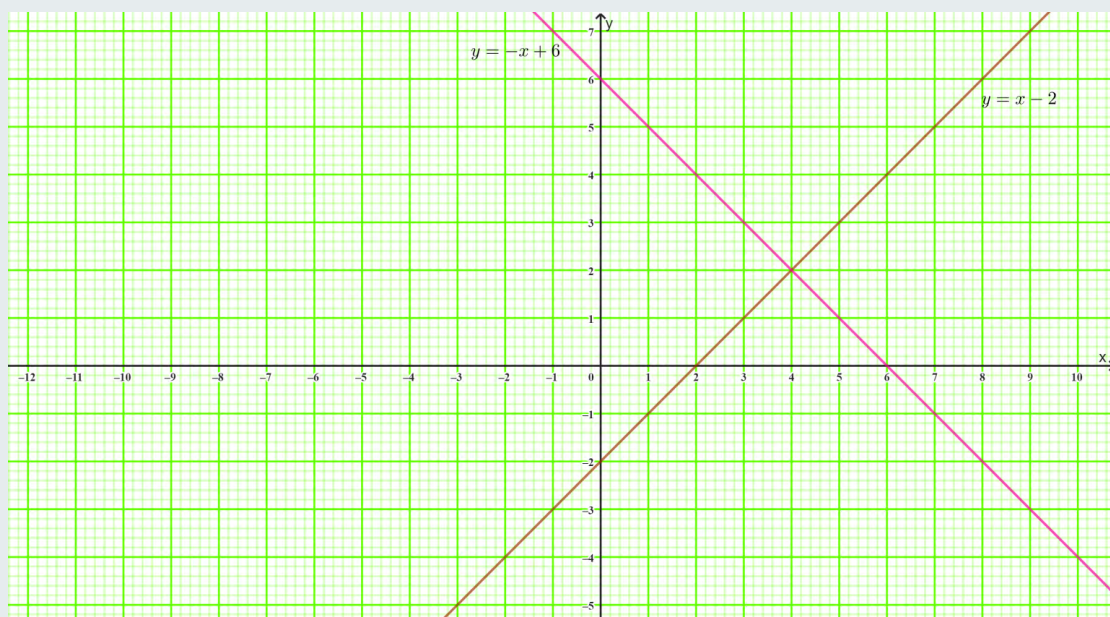


Figure 2.3: Graph of a simultaneous equation

1. What is the gradient of the line with the equation $x - y = 2$?
2. Which of the two lines has a negative gradient?
3. What is the coordinate of the point of intersection of the two lines?
4. Show that the point of intersection is truly the solution to the two systems of equations.

Solution

1. $x - y = 2$

Rewrite each equation in slope-intercept form (i.e., $y = mx + c$)

$$y = x - 2$$

Gradient = 1 as the coefficient of the x term in this form tells us the gradient.

2. The line $x + y = 6$ has a negative gradient, as $y = -x + 6$. The negative coefficient of x indicates the negative gradient.
3. The coordinate of the point of intersection of the two lines is $(4, 2)$
4. $(4, 2)$ means $x = 4, y = 2$
 For $x - y = 2$
 $4 - 2 = 2$ True
 For $x + y = 6$
 $4 + 2 = 6$ True

REAL-LIFE PROBLEMS INVOLVING SIMULTANEOUS EQUATIONS

To effectively tackle real-life problems involving simultaneous equations, use the following strategies:

1. Read the problem carefully to identify the quantities involved and what needs to be found.
2. Assign variables to the unknowns in the problem
3. Translate the problem into mathematical equations based on the relationships described in the problem.
4. Use methods such as substitution or elimination or graphical to solve the simultaneous equations.
5. Once you have found the values of your variables, interpret them in the context of the original problem.
6. For good practice and to confirm you have the correct solution, substitute your values back into the original equations to verify they satisfy both equations.

Activity 2.9: Real-life problem

Sarah buys 3 apples and 2 bananas for Gh¢7.50. John buys 2 apples and 3 bananas for Gh¢7.00. How much does one apple and one banana cost?

Solution

You need to represent the situation with a system of simultaneous linear equations.

Let:

x represent the cost of one apple.

y represent the cost of one banana

Step 1: Formulate the system of equations from the problem

Sarah buys 3 apples and 2 bananas for Gh¢7.50

$$3x + 2y = 7.50 \text{ (Equation 1)}$$

John buys 2 apples and 3 bananas for Gh¢7.00.

$$2x + 3y = 7 \text{ (Equation 2)}$$

Step 2: To eliminate x , we will multiply the first equation by 2 and the second equation by 3, making the coefficients of x equal to 6 in both equations.

Multiply Equation 1 by 2:

$$2(3x + 2y) = 2(7.50)$$

$$6x + 4y = 15 \text{ (Equation 3)}$$

Multiply Equation 2 by 3:

$$2x + 3y = 7$$

$$3(2x + 3y) = 3(7)$$

$$6x + 9y = 21 \text{ (Equation 4)}$$

Step 3: The coefficients of x are the same sign, so we subtract Equation 3 from Equation 4 to eliminate x .

$$(6x + 9y) - (6x + 4y) = 21 - 15$$

$$6x - 6x + 9y - 4y = 6$$

$$5y = 6$$

Step 4: Divide both sides by 5 to find y :

$$y = \frac{6}{5} = 1.20$$

The cost of one banana is Gh¢1.20

Step 5: Now that you have the cost of a banana, we can substitute $y = 1.20$ into one of the original equations to find x , the cost of an apple. Using equation 1:

$$3x + 2(1.20) = 7.50$$

$$3x + 2.40 = 7.50$$

Step 6: Subtract 2.40 from both sides of the equation to isolate x :

$$3x + 2.40 - 2.40 = 7.50 - 2.40$$

$$3x = 5.10$$

Step 7: Divide both sides by 3 to find x

$$x = \frac{5.10}{3} = 1.70$$

The cost of one apple is Gh¢1.70.

Activity 2.10: Real-life problem

A market woman in Accra sells plantains and tomatoes. She sells 2 plantains and 3 tomatoes for Gh¢12 and she sell 1 plantain and 1 tomato for Gh¢5. How much does 1 plantain cost and how much does 1 tomato cost?

Solution

Let:

x = cost of 1 plantain

y = cost of 1 tomato

$2x + 3y = 12$ (2 plantains and 3 tomatoes for GH¢12): Equation 1

$x + y = 5$ (1 plantain and 1 tomato for Gh¢5): Equation 2

Step 1: Rearrange equation 2 for x to be the subject:

$$x = 5 - y$$

Step 2: Substitute x into Equation 1:

$$2(5 - y) + 3y = 12$$

Step 3: Simplify:

$$10 - 2y + 3y = 12$$

$$10 + y = 12$$

$$y = 2$$

The cost of one tomato is Gh¢2.00

Step 4: Substitute y back into any of the original equation. Using equation 2:

$$x + 2 = 5$$

$$x = 3$$

The cost of one plantain is Gh¢3.00

Step 5: Confirm your values work and you have the correct solution.

Activity 2.11: Real-life problem

A farmer in the Northern Region sells maize and soybeans. He sells 2 bags of maize and 1 bag of soybeans for Gh¢150. If he sells 1 bag of maize for Gh¢40, how much does 1 bag of soybeans cost?

Solution

$2x + y = 150$ (2 bags of maize and 1 bag of soybeans for Gh¢150): Equation 1

$x = 40$ (1 bag of maize costs Gh¢40): Equation 2

Step 1: Substitute $x = 40$ into Equation 1:

$$2(40) + y = 150$$

Step 2: Simplify:

$$80 + y = 150$$

Step 3: Solve for y :

$$y = 150 - 80$$

$$y = 70$$

$x = 40$ (cost of 1 bag of maize)

$y = 70$ (cost of 1 bag of soybeans)

Therefore, 1 bag of soybeans costs Gh¢70.

Activity 2.12: Real-life problem

In the local market of **Chinderi**, a town in the **Oti Region**, the price of a pineapple is Gh¢2.00 and the price of a mango is Gh¢3.00. **Nana Okoegye Abraham**, the chief of Chinderi, went to the market and bought a total of 10 fruits, spending Gh¢24.00.

How many pineapples and how many mangoes did he buy?

Solution

Let:

x represent the number of pineapples

y represent the number of mangoes

$x + y = 10$ (total fruits): Equation 1

$2x + 3y = 24$ (total cost): Equation 2

Step 1: Rearrange Equation 1 for x

$$x = 10 - y$$

Step 2: Substitute x into Equation 2:

$$2(10 - y) + 3y = 24$$

Step 3: Simplify:

$$20 - 2y + 3y = 24$$

$$20 + y = 24$$

Step 4: Solve for y :

$$y = 4$$

Step 5: Substitute y back into Equation 1:

$$x + 4 = 10$$

$$x = 6$$

$x = 6$ (number of pineapples)

$y = 4$ (number of mangoes)

Therefore, Nana Okoegye Abraham bought 6 pineapples and 4 mangoes.

Using Software Demos to Graph Simultaneous Equations

Objective

You will learn how to use the Demos software on your phone to graph simultaneous equations with two variables. This activity will help you visualise solutions to equations and understand their graphical representations. Follow these steps to install and use the app with the guidance of your teacher.

Step 1: Download and Install Demos Software*For Android Users:*

1. Open the **Google Play Store** on your Android phone.
2. In the search bar, type “**Demos Graphing Calculator**” and select the app from the results.
3. Tap **Install** to download the app.
4. Once installed, open the app and allow any necessary permissions.

For Apple Users (iOS):

1. Open the **App Store** on your iPhone or iPad.
2. In the search field, type “**Demos Graphing Calculator**” and select the app from the results.
3. Tap **Get** and, if prompted, confirm with your Apple ID or passcode.
4. Once installed, open the app and allow any necessary permissions.

Step 2: Prepare for Graphing

With your teacher’s guidance, you will:

1. Open the Demos app on your phone.
2. Enter the simultaneous equations as instructed by your teacher.
3. Use the app’s graphing features to view the intersection points and understand solutions visually.

Note: Remember to bring your phone to class fully charged or have it ready at home during designated learning time. If you have questions about downloading, installing or using the software, ask your teacher for help.

EXTENDED READING

1. *Akrong Series: Core mathematics for Senior High Schools* New International Edition (Pages 196 – 201)
2. Aki – Ola series. *Core Mathematics for Senior High Schools in West Africa*, Millennium edition 5 (Pages 93– 102)
3. Baffour Asamoah, B. A. (2015). *Baffour BA series: Core mathematics*. Accra: Mega Heights, (Pages 438 - 458)

REVIEW QUESTIONS 2

1. Solve these simultaneous equations using the elimination method.
 - a. $5x + 2y = 14$
 $3x + y = 7$
 - b. $2x + 3y = 12$
 $x - 4y = -6$
 - c. $6x + y = 10$
 $4x - y = 10$
 - d. $7x + 3y = 20$
 $2x - 3y = -5$
2. Solve these simultaneous equations using the substitution method.
 - a. $x + 3y = 9$
 $2x - y = 4$
 - b. $2x + y = 5$
 $3x - 4y = 10$
 - c. $x - 2y = 7$
 $4x + y = 3$
 - d. $5x - y = 11$
 $x + y = 7$
3. Solve these pairs of simultaneous equations graphically
 - a. $2x + 5y = 9$ and $2x + 3y = 7$
 - b. $3x + 4y = 23$ and $2x - 4y = 2$
 - c. $7x + 2y = 33$ and $4x + 2y = 24$
4. A farmer sells two types of crops: maize and beans. The farmer sells maize at Gh¢ 4.00 per bowl and beans at Gh¢ 6.00 per bowl. If he sold a total of 30 bowls for Gh¢150.00, how many bags of maize and beans did he sell?
5. Ama and Kofi run a small food stall in Accra, selling two types of dishes: rice with chicken for Gh¢ 10 and jollof rice for Gh¢ 8. On a busy Saturday, they sold a total of 50 dishes and made Gh¢ 420 from their sales.

- a** Set up a system of simultaneous equations based on the information provided.
- b** Solve the system of equations to find out how many dishes of each type were sold that day.